Equivariant Jones Polynomials of periodic links

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Abstract

This paper continues the study of periodic links started in [Pol15]. It contains a study of the equivariant analogues of the Jones polynomial, which can be obtained from the equivariant Khovanov homology. In this paper we describe basic properties of such polynomials, show that they satisfy an analogue of the skein relation and develop a state-sum formula. The skein relation in the equivariant case is used to strengthen the periodicity criterion of Przytycki from [Prz89]. The state-sum formula is used to reproved the classical congruence of Murasugi from [Mur88].

1 Introduction

A link is periodic if it possesses certain rotational symmetry of finite order. To be more precise, periodic links, are invariant under some semi-free action of a cyclic group $\mathbb{Z}/n$ on S^3. Due to the resolution of the Smith Conjecture, see [MB84], this property can be restated in the following way. A link is n-periodic, if it is invariant under a rotation of $\mathbb{R}^3$ of order n. Additionally, we require the link to be disjoint from the axis of the rotation.

The relevance of periodic links stems from the fact, that according to [PS01], many finite order symmetries of 3-manifolds come from symmetries of their Kirby diagrams, i.e., if a 3-manifolds admits an action of a cyclic group of finite order with the fixed point homeomorphic to a circle, then it is a result of a Dehn surgery on a periodic link. Additionally, periodic links should give us an insight into the theory of skein modules of branched and unbranched covers.

Link polynomials of periodic links have been studied by many authors. For example [DL91, Mur71] study the Alexander polynomial of periodic links. The first paper, which studies the Jones polynomial of periodic links is [Mur88], where the author obtained a congruence which gives a relation between the Jones polynomial of a periodic link and a Jones polynomial of its quotient in terms of a certain congruence. This criterion proved to be very efficient in verifying whether a given link is periodic or not. In [Tra90], another periodicity criterion is given. This criterion gives another congruence for the Jones polynomial. This criterion was later generalized in [Prz89, Yok91]. A survey of the results on the Jones polynomials of periodic links can be found in [Prz04]

This paper is a continuation of [Pol15], where the author developed a link homology theory, called the Khovanov homology, which is an adaptation of the Khovanov homology to the equivariant setting of periodic links. In this paper, instead of studying the homology, we study analogues of the Jones polynomial, which can be obtained from the equivariant Khovanov homology. We utilize

the properties of the homology theory described in [Pol15] to obtain properties of the equivariant Jones polynomials, which are further used to obtain some periodicity criteria for the Jones polynomial.

In [Pol15] the rational equivariant Khovanov homology of an n-periodic link L was defined as a triply-graded $\mathbb{Q}$-vector space

$$\mathrm{Kh}^{*,*,*}_{\mathbb{Z}/n}(L;\mathbb{Q}),$$

where the third grading is supported only for dimensions d such that $d \mid n$ and $d > 0$. In fact, for any $d \mid n$, the vector space

$$\mathrm{Kh}^{*,*,d}_{\mathbb{Z}/n}(L;\mathbb{Q})$$

is a vector space over the larger field $\mathbb{Q}[\xi_d]$, i.e. the d-th cyclotomic field. This structure is taken into account in the definition of the equivariant Jones polynomials.

$$\mathrm{J}_{n,d}(L) = \sum_{i,j} t^i q^j \dim_{\mathbb{Q}[\xi_d]} \mathrm{Kh}^{i,j,d}_{\mathbb{Z}/n}(L;\mathbb{Q}).$$

Theorem 3.2 summarizes basic properties of these polynomials.

When p is a prime and we study p^n-periodic links, it is better to consider the following difference Jones polynomials

$$\mathrm{DJ}_{n,s}(L) = \mathrm{J}_{p^n,p^s}(L) - \mathrm{J}_{p^n,p^{s+1}}(L),$$

for $0 \leq s \leq n$. It turns out, that these polynomials have much better properties than the equivariant Jones polynomials. The first main result of this paper is the following theorem.

Theorem 3.6. The difference Jones polynomials have the following properties

1. DJ_0 satisfies the following version of the skein relation

$$q^{-2p^n}\,\mathrm{DJ}_{n,0}\left(\ldots\right) - q^{2p^n}\,\mathrm{DJ}_{n,0}\left(\ldots\right) =$$
$$= \left(q^{-p^n} - q^{p^n}\right)\mathrm{DJ}_{n,0}\left(\ldots\right),$$

   where $\ldots$, $\ldots$ and $\ldots$ denote the orbit of positive, negative and orientation preserving resolutions of crossing, respectively.

2. for any $0 \leq s \leq n$, DJ_s satisfies the following congruences

$$q^{-2p^n}\,\mathrm{DJ}_{n,n-s}\left(\ldots\right) - q^{2p^n}\,\mathrm{DJ}_{n,n-s}\left(\ldots\right) \equiv$$
$$\equiv \left(q^{-p^n} - q^{p^n}\right)\mathrm{DJ}_{n,n-s}\left(\ldots\right) \pmod{q^{p^s} - q^{-p^s}}.$$

In other words, difference Jones polynomials satisfy an analogue of the skein relation of the classical Jones polynomial. The above theorem can be used to easily recover the periodicity criterion from [Prz89]. However, if we use one other property of the equivariant Khovanov homology, we can strengthen this criterion considerably.

Theorem 3.9. Suppose that L is a p^n-periodic link and for all i,j we have $\dim_{\mathbb{Q}} \mathrm{Kh}^{i,j}(L;\mathbb{Q}) < \varphi(p^s)$, then the following congruence holds

$$\mathrm{J}(L)(q) \equiv \mathrm{J}(L)(q^{-1}) \pmod{\mathcal{I}_{p^n,s}},$$

where $\mathcal{I}_{p^n,s}$ is the ideal generated by the following monomials

$$q^{p^n} - q^{-p^n}, p\left(q^{p^{n-1}} - q^{-p^{n-1}}\right), \ldots, p^{s-1}\left(q^{p^{n-s+1}} - q^{-p^{n-s+1}}\right)$$

and J denotes the unreduced Jones polynomial.

Example 1.1. *Consider the* 10_{61} *knot from the Rolfsen table* [Rol90]*. If we are interested in the symmetry of order* 5*, then according to SAGE* [S$^+$14]*, the following congruence holds*

$$\mathrm{J}(10_{61})(q) - \mathrm{J}(10_{61})(q^{-1}) \equiv 0 \pmod{q^5 - q^{-5}, 5(q - q^{-1})}.$$

Hence, Przytycki's Theorem does not obstruct 10_{61} *to have symmetry of order* 5*. However, if we notice that, as depicted on Figure 1, the dimension of* $\mathrm{Kh}^{i,j}(10_{61})$ *is always smaller that* $\varphi(5) = 4$ *and apply Theorem 3.9 we obtain*

$$\mathrm{J}(10_{61})(q) - \mathrm{J}(10_{61})(q^{-1}) \not\equiv 0 \pmod{q^5 - q^{-5}}$$

Consequently 10_{61} *is not* 5*-periodic.*

The second main theorem of this papers gives a state-sum formula for the difference Jones polynomials. This formula shows that in order to obtain the difference polynomial $\mathrm{DJ}_{n,n-m}(L)$ we need to consider only Kauffman states $s \in \mathcal{S}^{p^v}(D)$, i.e., those Kauffman states, which inherit a symmetry of order p^v from L, for $m \le v \le n$.

Theorem 4.1. Let D be a p^n-periodic diagram of a link and let $0 \le m \le n$. Under this assumptions, the following equality holds.

$$\mathrm{DJ}_{n,n-m}(D) = (-1)^{n_-(D)} q^{n_+(D)-2n_-(D)} \sum_{m \le v \le n} \sum_{s \in \mathcal{S}^{p^v}(D)} (-q)^{r(s)} \, \mathrm{DJ}_{v,v-m}(s).$$

For a Kauffman state s we write $r(s) = r$ if $s \in \mathcal{S}_r(D)$, compare Definition 2.4.

The state sum formula for $\mathrm{DJ}_{n,0}(L)$ is used to reprove the congruence from [Mur88].

The paper is organized as follows. Section 2 contains a summary of results about the equivariant Khovanov homology, which are needed in this paper. In section 3 we describe the basic properties of the equivariant Jones polynomials, and difference polynomials. Further, we use Theorem 3.6 to generalize Przytycki's periodicity criterion. Section 4 contains the discussion of the state sum formula and its implications. Section 5 contains the proofs of Theorem 3.6 and Theorem 4.1.

Convention In the remainder part of this article we adopt the convention that all links are oriented unless stated otherwise. Furthermore, we always use the unreduced Jones polynomial.

j \ i	-4	-3	-2	-1	0	1	2	3	4	5	6
17											1
15										1	
13									2	1	
11								2	1		
9							3	2			
7						2	2				
5					2	3					
3				3	3						
1				1							
-1		1	3								
-3											
-5	1										

Kh

Figure 1: Ranks of $\mathrm{Kh}^{i,j}(10_{61})$ according to [See14].

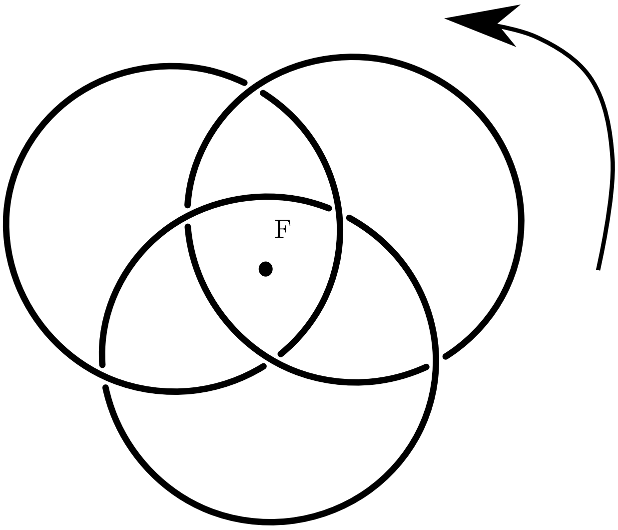

Figure 2: Borromean rings are 3-periodic. The fixed point axis F is marked with a dot.

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2 Equivariant Khovanov homology

This section surveys the results from [Pol15], which will be needed in this paper. We only focus on the rational equivariant homology, since our main focus is on the equivariant analogues of the Jones polynomial.

Let us start with the recollection of the definition of a periodic link.

Definition 2.1. *Let n be a positive integer, and let L be a link in S^3. We say that L is n-periodic, if there exists an action of the cyclic group of order n on S^3 satisfying the following conditions.*

1. *The fixed point set, denoted by F, is the unknot.*
2. *L is disjoint from F.*
3. *L is a $\mathbb{Z}/n$-invariant subset of S^3.*

Example 2.2. *Borromean rings provide an example of a 3-periodic link. The symmetry is visualized on Figure 2. The dot marks the fixed point axis.*

Example 2.3. *Torus links constitute an infinite family of periodic links. In fact, according to* [Mur71], *the torus link $T(m, n)$ is d-periodic if, and only if, d divides either m or n.*

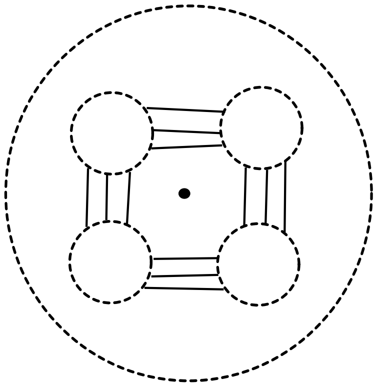

Figure 3: 4-periodic planar diagram.

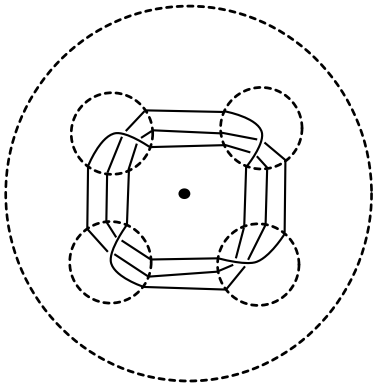

Figure 4: Torus knot $T(3,4)$ as a 4-periodic knot obtained from the planar diagram from Figure 3

Periodic diagrams of periodic links can be described in terms of planar algebras. For the definition of planar algebras see [BN05]. Take an n-periodic planar diagram D_n with n input disks, like the one on Figure 3. Choose a tangle T which possesses enough endpoints, and glue n copies of T into the input disks of D_n. In this way, we obtain a periodic link whose quotient is represented by an appropriate closure of T. See Figure 4 for an example.

Let D be an n-periodic diagram of an n periodic link. The Kauffman states of D can be naturally divided into several families according to the type of symmetry they inherit from D.

Definition 2.4. 1. *Let $\mathcal{S}_r(D)$ denote the set of Kauffman states of D which were obtained by resolving exactly r crossings with the* 1*-smoothing.*

2. *For $d \mid n$, let $\mathcal{S}^d(D)$ denote the set of Kauffman states which inherit a symmetry of order d from the symmetry of D, that is Kauffman states of the form*

$$D_n(T_1, \ldots, T_{\frac{n}{d}}, T_1, \ldots, T_{\frac{n}{d}}, \ldots, T_1, \ldots, T_{\frac{n}{d}}),$$

where $T_1, \ldots, T_{\frac{n}{d}}$ are distinct resolutions of T.

3. *For a Kauffman state s, write* $\mathrm{Iso}_D(s) = \mathbb{Z}/d$ *if, and only if* $s \in \mathcal{S}^d(D)$.

4. *Define* $\mathcal{S}^d_r(D) = \mathcal{S}^d(D) \cap \mathcal{S}_r(D)$.

5. *Define* $\overline{\mathcal{S}}^d_r(D)$ *to be the quotient of* $\mathcal{S}^d_r(D)$ *by the action of* $\mathbb{Z}/n$.

Remark 2.5. *If* $\mathcal{S}^d_r(D)$ *is non-empty, then* $d \mid \gcd(n, r)$.

Analysis of the Khovanov complex $\mathrm{CKh}(D; \mathbb{Q})$ for an n-periodic link diagram D shows that the symmetry of the diagram can be lifted to an action of the cyclic group $\mathbb{Z}/n$ on the Khovanov complex. This leads to the following conclusion.

Proposition 2.6. *If D is a periodic link diagram, then* $\mathrm{CKh}(D; \mathbb{Q})$ *is a complex of graded* $\mathbb{Q}[\mathbb{Z}/n]$*-modules.*

This leads to the definition of the equivariant Khovanov homology.

Definition 2.7. *Define the rational equivariant Khovanov homology of an n-periodic diagram D to be the following triply-graded module, for which the third grading is supported only for $d \mid n$.*

$$\begin{aligned}\mathrm{Kh}^{*,*,d}_{\mathbb{Z}/n}(D; \mathbb{Q}) &= H^{*,*}\left(\mathrm{Hom}_{\mathbb{Q}[\mathbb{Z}/n]}\left(\mathbb{Q}[\xi_d], \mathrm{CKh}(D; \mathbb{Q})\right)\right) \\ &\cong H^{*,*}\left(\mathbb{Q}[\xi_d] \otimes_{\mathbb{Q}[\mathbb{Z}/n]} \mathrm{CKh}(D; \mathbb{Q})\right),\end{aligned}$$

where $\mathrm{CKh}(D; \mathbb{Q})$ *is the Khovanov complex of D with rational coefficients. It is worth to notice, that since* $\mathrm{CKh}(D)$ *is a complex of graded modules, Ext groups become also naturally graded, provided that we regard* $\mathbb{Z}[\xi_d]$ *as a graded module concentrated in degree* 0.

The equivariant Khovanov homology depends only on the equivariant isotopy class of the link represented by the diagram i.e. we allow deformations of the diagram which commute with the action of the cyclic group. Such deformations can be realized as a composition of equivariant Reidemeister moves.

Theorem 2.8. *Equivariant Khovanov homology groups are invariants of periodic links, that is they are invariant under equivariant Reidemeister moves.*

Semi-simplicity of the group algebra has a number of implications regarding the structure of the rational equivariant Khovanov homology.

Proposition 2.9. *Let D be an n-periodic link diagram.*

1. *The equivariant Khovanov homology can be computed in the following way*

$$\begin{aligned}\mathrm{Kh}^{*,*,d}_{\mathbb{Z}/n}(D; \mathbb{Q}) &\cong \mathbb{Q}[\xi_d] \otimes_{\mathbb{Q}[\mathbb{Z}/n]} \mathrm{Kh}^{*,*}(D; \mathbb{Q}) \\ &\cong e_d \, \mathrm{Kh}^{*,*}(D; \mathbb{Q}),\end{aligned}$$

where $e_d \in \mathbb{Q}[\mathbb{Z}/n]$ is a central idempotent, which acts as an identity on an irreducible summand of isomorphic to $\mathbb{Q}[\xi_d]$ and annihilates the complementary summand.

2. *There exists a decomposition of* $\mathbb{Q}[\mathbb{Z}/n]$*-modules*

$$\mathrm{Kh}^{*,*}(D; \mathbb{Q}) = \bigoplus_{d \mid n} \mathrm{Kh}^{*,*,d}_{\mathbb{Z}/n}(D; \mathbb{Q}).$$

3. *If for any i, j*

$$\dim \mathrm{Kh}^{i,j}(D) < \varphi(d),$$

where φ is the Euler's totient functions, then

$$\mathrm{Kh}^{*,*,d}_{\mathbb{Z}/n}(D;\mathbb{Q}) = 0.$$

Proof. The first two statements are easy consequences of the existence of the Wedderburn decomposition of the group algebra and of the Schur's Lemma. The third one follows from the second, because $\mathrm{Kh}^{*,*,d}_{\mathbb{Z}/n}(D;\mathbb{Q})$ is a $\mathbb{Q}[\xi_d]$ vector spaces and

$$\dim \mathbb{Q}[\xi_d] = \varphi(d).$$

□

As it was shown in [Pol15] these properties can be used to determine the equivariant Khovanov homology in some cases.

Corollary 2.10. *Let $T(n,2)$ be the torus link. Let $d > 2$ be a divisor of n. According to Example 2.3, $T(n,2)$ is d-periodic. Let $d' > 2$ and $d' \mid d$.*

$$\mathrm{Kh}^{*,*,d'}_{\mathbb{Z}/d}(T(n,2);\mathbb{Q}) = 0.$$

Proof. Indeed, because according to [Kho00, Prop. 35] for all i, j we have

$$\dim_{\mathbb{Q}} \mathrm{Kh}^{i,j}(T(n,2);\mathbb{Q}) \leq 1$$

and $\varphi(d') > 1$ if $d' > 2$. □

Corollary 2.11. *Let $\gcd(3,n) = 1$. The 3-equivariant Khovanov homology $\mathrm{Kh}^{*,*,3}_{\mathbb{Z}/3}(T(n,3);\mathbb{Q})$ of the torus knot $T(n,3)$ vanishes.*
If $d > 2$ divides n, $d' > 2$ and $d' \mid d$, then $\mathrm{Kh}^{*,*,d'}_{\mathbb{Z}/d}(T(n,3);\mathbb{Q}) = 0$.

Proof. Indeed, because [Tur08, Thm 3.1] implies that for all i, j we have

$$\dim_{\mathbb{Q}} \mathrm{Kh}^{i,j}(T(n,3);\mathbb{Q}) \leq 1,$$

provided that $\gcd(3,n) = 1$. □

Let T_{kp^n+f} be a crossingless diagram of the trivial link with $kp^n + f$ components, for a prime p and non-negative integers k, n, f. This link is p^n-periodic in such a way that its components can be divided into two families. The first family contains kp^n components, which are freely permuted, and the second one contains the remainder components which are linked with the fixed point axis and are rotated by the diffeomorphism generating the action of the cyclic group. In order to state the result of the computation of the equivariant Khovanov homology of T_{kp^n+f}, let us introduce some notation.

Definition 2.12. *Define a sequence of Laurent polynomials i.e. elements of the ring* $\mathbb{Z}\left[q, q^{-1}\right]$.

$$\begin{aligned}\mathcal{P}_0(q) &= q + q^{-1}\\ \mathcal{P}_n(q) &= \frac{1}{p^n} \sum_{\substack{1 \leq k \leq p^n - 1\\ \gcd(k,p^n)=1}} \binom{p^n}{k} q^{2k-p^n} +\\ &\quad + \frac{1}{p^n} \sum_{1 \leq s < n} \sum_{\substack{1 \leq k \leq p^n - 1\\ \gcd(k,p^n)=p^s}} \left(\binom{p^n}{k} - \binom{p^{n-s}}{k'} \right) q^{2k-p^n},\end{aligned}$$

where $k' = k/p^s$ *and* $n \geq 1$.

Definition 2.13. *If* M^* *is a graded* $\mathbb{Q}$ *vector space, then define its quantum dimension as in* [Tur08].

$$\operatorname{qdim} M^* = \sum_i q^i \dim M^i \in \mathbb{Z}\left[q, q^{-1}\right].$$

Definition 2.14. *Let* $M_s^{k,f}$ *be a graded* $\mathbb{Q}$ *vector space whose quantum dimension satisfies the following equality*

$$\begin{aligned}&\operatorname{qdim} M_s^{k,f} =\\ &= (q+q^{-1})^f \sum_{\ell=1}^{k} p^{s\cdot(\ell-1)} \mathcal{P}_s(q^{p^{n-s}})^{\ell} \sum_{\substack{0 \leq i_0, \ldots, i_{s-1} \leq k\\ i_0 + \ldots + i_{s-1} = k-\ell}} \prod_{j=0}^{s-1} \left(p^j \mathcal{P}_j(q^{p^{n-j}}) \right)^{i_j}.\end{aligned}$$

Proposition 2.15. *The rational Khovanov homology of the trivial link* T_{kp^n+f}*, for some prime p, which possesses k free orbits of components and f fixed circles, is given by the following formula*

$$\operatorname{Kh}^{0,*,p^{n-u}}_{\mathbb{Z}/p^n}(T_{kp^n+f}; \mathbb{Q}) = \bigoplus_{s=n-u}^{n} (M_s^{k,f})^{\varphi(p^{n-u})}.$$

[Pol15] contains also a construction of a spectral sequence converging to the equivariant Khovanov homology, which is a substitute for the long exact sequence of Khovanov homology. First, however, we need to introduce some notation.

Start with a link L and its n-periodic diagram D. Choose a subset of crossings $X \subset \operatorname{Cr} D$.

Definition 2.16. *Let* $\alpha\colon \operatorname{Cr}(D) \to \{0, 1, x\}$ *be a map.*

1. *If* $i \in \{0, 1, x\}$ *define* $|\alpha|_i = \#\alpha^{-1}(i)$.
2. *Define the support of* α *to be* $\operatorname{supp} \alpha = \alpha^{-1}(\{0, 1\})$.
3. *Define also the following family of maps*

$$\mathcal{B}_k(X) = \{\alpha\colon \operatorname{Cr}(D) \to \{0, 1, x\} \quad | \quad \operatorname{supp} \alpha = X, \quad |\alpha|_1 = k\}.$$

4. *Denote by* D_α *the diagram obtained from* D *by resolving crossings from* $\alpha^{-1}(0)$ *by* 0- *and from* $\alpha^{-1}(1)$ *by* 1*-smoothing.*

Definition 2.17. *If* D *is a link diagram,* $X \subset \operatorname{Cr} D$ *and* $\alpha \in \mathcal{D}_k(X)$*, for some* k*, define*

$$c(D_\alpha) = n_-(D_\alpha) - n_-(D),$$

where $n_-(D_\alpha)$ *denotes the number of negative crossings of* D_α*.*

Definition 2.18. *Let* $M^{*,*}$ *be a bi-graded module. For* $m, n \in \mathbb{Z}$ *define a new bi-graded module* $M[n]\{m\}$ *in the following way.*

$$M[n]\{m\}^{i,j} = M^{i-n,j-m}.$$

The construction of the spectral sequence starts with a choice of a single orbit of crossings of a periodic diagram D. To preserve the symmetry, we have to consider all resolutions of the crossings from the chosen orbit. Khovanov complex of D can be made into a bicomplex whose columns are expressible in terms of the Khovanov complexes of the diagrams obtained from D by resolving only crossings from the chosen orbit. This leads to the following spectral sequence.

Theorem 2.19. *Let* L *be a* p^n*-periodic link, where* p *is an odd prime, and let* $X \subset \operatorname{Cr} D$ *consists of a single orbit. Under this assumption, for any* $0 \le s \le n$ *there exists a spectral sequence* $\{{}_{p^{n-s}}E_r^{*,*}, d_r\}$ *of graded modules converging to* $\operatorname{Kh}^{*,*,p^{n-s}}_{\mathbb{Z}/p^n}(D;\mathbb{Q})$ *with*

$$\begin{aligned}
{}_{p^{n-s}}E_1^{0,j} &= \operatorname{Kh}^{j,*,p^{n-s}}_{\mathbb{Z}/p^n}(D_{\alpha_0};\mathbb{Q})[c(D_{\alpha_0})]\{q(\alpha_0)\},\\
{}_{p^{n-s}}E_1^{p^n,j} &= \operatorname{Kh}^{j,*,p^{n-s}}_{\mathbb{Z}/p^n}(D_{\alpha_1};\mathbb{Q})[c(D_{\alpha_1})]\{q(\alpha_1)\},\\
{}_{p^{n-s}}E_1^{i,j} &= \bigoplus_{0\le v\le u_i} \bigoplus_{\alpha\in\overline{\mathcal{B}}_i^{p^v}(X)} \operatorname{Kh}^{j,*,k(v,s)}_{\mathbb{Z}/p^v}(D_\alpha;\mathbb{Q})[c(D_\alpha)]\{q(\alpha)\}^{\ell(v,s)}
\end{aligned}$$

for $0 < i < p^n$*. Above we used the following notation* $i = p^{u_i}g$*, where* $\gcd(p,g) = 1$ *and* α_0*,* α_1 *are the unique elements of* $\mathcal{B}_0(X)$ *and* $\mathcal{B}_{p^n}(X)$*, respectively, and*

$$\begin{aligned}
q(\alpha) &= i + 3c(D_\alpha) + p^n,\\
k(s,v) &= \begin{cases} 1, & v \le s,\\ p^{v-s}, & v > s,\end{cases}\\
\ell(s,v) &= \begin{cases} \varphi(p^{n-s}), & v \le s,\\ p^{n-v}, & v > s,\end{cases}
\end{aligned}$$

3 Equivariant Jones polynomials

Analogously as in the classical case, we can define the equivariant Jones polynomials of a periodic link.

Definition 3.1. *Let* L *be an* n*-periodic link. For* $d \mid n$ *define a* d*-th equivariant Khovanov polynomial by*

$$\operatorname{KhP}_{n,d}(L)(t,q) = \sum_{i,j} t^i q^j \dim_{\mathbb{Q}[\xi_d]} \operatorname{Kh}^{i,j,d}(L;\mathbb{Q})$$

and d-th equivariant Jones polynomial

$$\mathrm{J}_{n,d}(L)(q) = \mathrm{KhP}_{n,d}(L)(-1, q).$$

The next theorem describes basic properties of equivariant Jones polynomials.

Theorem 3.2. *Let L be an n-periodic link and let $d \mid n$.*

1. *Equivariant Khovanov and Jones polynomials are invariants of periodic links i.e. they are invariant under Reidemeister moves.*
2. *If $\mathrm{J}(L)$ denotes the ordinary unreduced Jones polynomial, then the following equality holds.*
$$\mathrm{J}(L) = \sum_{d|n} \phi(d)\, \mathrm{J}_{n,d}(L),$$
where ϕ denotes the Euler's totient function.
3. *If $d \mid n$ and for all i, j we have $\dim_{\mathbb{Q}} \mathrm{Kh}^{i,j}(L; \mathbb{Q}) < \varphi(d)$, then*
$$\begin{aligned} \mathrm{KhP}_{n,d}(L) &= 0, \\ \mathrm{J}_{n,d}(L) &= 0. \end{aligned}$$

Proof. The first part of the theorem follows from theorem 2.8. The remaining two are consequences of Proposition 2.9. □

From now on we assume that all links are p^n-periodic, for p an odd prime and $n > 0$.

Definition 3.3. *Suppose that D is a p^n-periodic link diagram. Define the difference Jones polynomials*

$$\mathrm{DJ}_{n,s}(D) = \mathrm{J}_{p^n,p^s}(D) - \mathrm{J}_{p^n,p^{s+1}}(D)$$

for $0 \leq s \leq n$.

Corollary 3.4. *The following equality holds.*

$$\mathrm{J}(D) = \sum_{s=0}^{n} p^s\, \mathrm{DJ}_{n,s}(D)$$

Proof. Proof follows from theorem 3.2 and a simple fact that

$$\phi(p^s) = p^s - p^{s-1}.$$

□

Example 3.5. *Let $T_{k \cdot p^n + f}$ be as in Proposition 2.15. Proposition 2.15 implies that the equivariant and difference Jones polynomials of $T_{k \cdot p^n + f}$ can be expressed in terms of polynomials $\mathcal{P}_s$ from Definition 2.12.*

$$\begin{aligned} \mathrm{J}_{p^n,p^{n-u}}(T_{k \cdot p^n + f}) &= \sum_{s=n-u}^{n} \operatorname{qdim} M_s^{k,f}, \\ \mathrm{DJ}_{n,n-u}(T_{k \cdot p^n + f}) &= \operatorname{qdim} M_{n-u}^{k,f}. \end{aligned}$$

One of the most important properties of the Jones polynomials is the skein relation

$$q^{-2}\,\mathrm{J}\left(\;\right) - q^{2}\,\mathrm{J}\left(\;\right) = \left(q^{-1} - q\right)\mathrm{J}\left(\;\right).$$

The skein relation is a consequence of the long exact sequence of Khovanov homology. Spectral sequences from Theorem 2.19 can be utilized to obtain the following analogue of the skein relation for the difference Jones polynomials.

Theorem 3.6. *The difference Jones polynomials have the following properties*

1. DJ_0 *satisfies the following version of the skein relation*

$$q^{-2p^n}\,\mathrm{DJ}_{n,0}\left(\;\ldots\;\right) - q^{2p^n}\,\mathrm{DJ}_{n,0}\left(\;\ldots\;\right) = \\ = \left(q^{-p^n} - q^{p^n}\right)\mathrm{DJ}_{n,0}\left(\;\ldots\;\right),$$

where $\ldots$, $\ldots$ *and* $\ldots$ *denote the orbit of positive, negative and orientation preserving resolutions of crossing, respectively.*

2. *for any* $0 \leq s \leq n$, DJ_s *satisfies the following congruences*

$$q^{-2p^n}\,\mathrm{DJ}_{n,n-s}\left(\;\ldots\;\right) - q^{2p^n}\,\mathrm{DJ}_{n,n-s}\left(\;\ldots\;\right) \equiv \\ \equiv \left(q^{-p^n} - q^{p^n}\right)\mathrm{DJ}_{n,n-s}\left(\;\ldots\;\right) \pmod{q^{p^s} - q^{-p^s}}.$$

Remark 3.7. *We defer the proof of Theorem 3.6 to Section 5.*

The above theorem has a number of consequences regarding the Jones polynomial of a periodic link. For example, it enables us to write down a few criteria for the periodicity of a link in terms of its Jones polynomial. One such example was given by J.H. Przytycki in [Prz89].

Theorem 3.8. *Suppose that L is a p^n-periodic link. Then the following congruence holds*

$$\mathrm{J}(L)(q) \equiv \mathrm{J}(L)(q^{-1}) \pmod{\mathcal{I}_{p^n}},$$

where $\mathcal{I}_{p^n}$ is an ideal generated by the following monomials

$$p^n, p^{n-1}\left(q^p - q^{-p}\right), \ldots, p\left(q^{p^{n-1}} - q^{p^{-n-1}}\right), q^{p^n} - q^{-p^n}.$$

Proof. Notice that

$$\mathrm{J}\left(\;\ldots\;\right) - \mathrm{J}\left(\;\ldots\;\right) \equiv \\ q^{2p^n}\,\mathrm{J}\left(\;\ldots\;\right) - q^{-2p^n}\,\mathrm{J}\left(\;\ldots\;\right) \equiv \\ \sum_{s=0}^{n} p^{n-s}\left(q^{2p^n}\,\mathrm{DJ}_{n,n-s}\left(\;\ldots\;\right) - q^{-2p^n}\,\mathrm{DJ}_{n,n-s}\left(\;\ldots\;\right)\right) \equiv \\ \equiv 0 \pmod{\mathcal{I}_{p^n}}.$$

Hence, switching crossings from a single orbit does not change the Jones polynomial modulo $\mathcal{I}_{p^n}$. Since we can pass from L to its mirror image $L^!$ by switching one orbit at at time, it follows that

$$\mathrm{J}(L) \equiv \mathrm{J}(L^!) \pmod{\mathcal{I}_{p^n}}.$$

Taking into account the relation between the Jones polynomials of L and $L^!$

$$\mathrm{J}(L^!)(q) = \mathrm{J}(L)(q^{-1})$$

concludes the proof ☐

The above theorem can be considerably strengthened with the aid of Theorem 3.2.

Theorem 3.9. *Suppose that L is a p^n-periodic link and for all i, j we have $\dim_{\mathbb{Q}} \mathrm{Kh}^{i,j}(L;\mathbb{Q}) < \varphi(p^s)$, then the following congruence holds*

$$\mathrm{J}(L)(q) \equiv \mathrm{J}(L)(q^{-1}) \pmod{\mathcal{I}_{p^n,s}},$$

where $\mathcal{I}_{p^n,s}$ is the ideal generated by the following monomials

$$q^{p^n} - q^{-p^n}, p\left(q^{p^{n-1}} - q^{-p^{n-1}}\right), \ldots, p^{s-1}\left(q^{p^{n-s+1}} - q^{-p^{n-s+1}}\right).$$

Proof. First notice that Theorem 3.2 implies that for $s' \geq s$ the equivariant Jones polynomials $\mathrm{J}_{n,p^{s'}}(L)$ vanish. Therefore $\mathrm{DJ}_{s'} = 0$. Corollary 3.4 implies that

$$\mathrm{J}(L) = \sum_{i=0}^{s-1} p^i \, \mathrm{DJ}_i \, .$$

Now argue as in the proof of the previous Theorem to obtain the desired result. ☐

4 State-sum formula

One of the most important properties of the Jones polynomial is that it can be written as a certain sum over all Kauffman states obtained from the given diagram. This point of view on the Jones polynomial was pioneered in [Kau87]. In particular, if D is a link diagram, then the Jones polynomial of the corresponding links expands into the following sum

$$\mathrm{J}(D) = (-1)^{n_-(D)} q^{n_+(D) - 2n_-(D)} \sum_{s \in \mathcal{S}(D)} (-q)^{r(s)} (q + q^{-1})^{k(s)},$$

where $k(s)$ denotes the number of components of the Kauffman state s and $r(s)$ denotes the number of 1-smoothings used to get the Kauffman state s, see [Tur06, Lect. 1]. As it turns out, it is possible to develop analogous expansions for the difference Jones polynomials.

Theorem 4.1. *Let D be a p^n-periodic diagram of a link and let $0 \leq m \leq n$. Under this assumptions, the following equality holds.*

$$\mathrm{DJ}_{n,n-m}(D) = (-1)^{n_-(D)} q^{n_+(D) - 2n_-(D)} \sum_{m \leq v \leq n} \sum_{s \in \mathcal{S}^{p^v}(D)} (-q)^{r(s)} \, \mathrm{DJ}_{v,v-m}(s).$$

For a Kauffman state s we write $r(s) = r$ if $s \in \mathcal{S}_r(D)$, compare Definition 2.4.

Remark 4.2. *The proof of the above Theorem is deferred to Section 5.*

An application of the state sum formula for the difference Jones polynomials is concerned with the following classical periodicity criterion of Murasugi [Mur88].

Theorem 4.3. *Let D be a p^n-periodic link diagram. Let D_* denote the quotient diagram.*

$$\mathrm{J}(D) \equiv \mathrm{J}(D_*)^{p^n} \pmod{p, (q+q^{-1})^{\alpha(D)(p^n-1)} - 1},$$

where

$$\alpha(D) = \begin{cases} 1, & 2 \nmid \mathrm{lk}(D, F), \\ 2, & 2 \mid \mathrm{lk}(D, F). \end{cases}$$

Above, F denotes the fixed point set.

Proof. Let us analyze the relation between $\mathrm{DJ}_{n,0}(D)$ and $\mathrm{J}(D_*)$.

Proposition 4.4. *The following congruence holds.*

$$\mathrm{DJ}_{n,0}(D) \equiv \mathrm{J}(D_*)^{p^n} \pmod{p, (q+q^{-1})^{\alpha(D)(p^n-1)} - 1}.$$

Proof. Notice that the state formula for $\mathrm{DJ}_{n,0}(D)$ involves only Kauffman states which inherit $\mathbb{Z}/p^n$-symmetry. Such Kauffman states correspond bijectively to the Kauffman states of the quotient diagram.

Let s be a Kauffman state obtained from D such that $\mathrm{Iso}(s) = \mathbb{Z}/p^n$. Assume that s consists of k free orbits and f fixed circles. Thus, according to Example 3.5 the Kauffman state contributes

$$\begin{aligned} &(-1)^{n_-(D)+r(s)} q^{n_+(D)-2n_-(D)+r(s)} (q^{p^n} + q^{-p^n})^k (q+q^{-1})^f \equiv \\ &\equiv (-1)^{n_-(D)+r(s)} q^{n_+(D)-2n_-(D)+r(s)} (q+q^{-1})^{kp^n+f} \pmod{p}. \end{aligned}$$

to the state sum for $\mathrm{DJ}_{n,0}$. The quotient Kauffman state s_* consists of $k+f$ components. Hence, it contributes

$$(-1)^{p^n(n_-(D_*)+r(s_*))} q^{p^n(n_+(D_*)-2n_-(D_*)+r(s_*))} (q+q^{-q})^{p^n(k+f)}$$

to $\mathrm{J}(D_*)^{p^n}$ mod p. The difference of both contributions is divisible by

$$(q+q^{-1})^{f(p^n-1)} - 1.$$

Since

$$f \equiv \mathrm{lk}(D, F) \pmod{2},$$

the proposition follows. □

We can conclude the proof once we note that

$$\mathrm{J}(D) \equiv \mathrm{DJ}_{n,1}(D) \pmod{p}$$

by Corollary 3.4. □

5 Proofs

This section is entirely devoted to the presentation of proofs of Theorems 3.6 and 4.1. These two proofs are very alike in principle, therefore we only perform detailed calculations in the first proof, because the very same calculations are present in the second proof as well.

5.1 Proof of Theorem 3.6

Let us begin with a definition.

Definition 5.1. *Let $\{E_s^{*,*}, d_s\}$ be a spectral sequence of graded finite-dimensional $\mathbb{F}$-modules, where $\mathbb{F}$ is a field, converging to some doubly-graded finite-dimensional $\mathbb{F}$-module $H^{*,*}$. Suppose that the spectral sequence collapses at some finite stage. Define the Poincaré polynomial of the E_s page to be the following polynomial.*

$$P(E_s)(t,q) = \sum_{i,j} t^{i+j} \operatorname{qdim}_{\mathbb{F}} E_s^{i,j}$$

Poincaré polynomial admits the following decomposition

$$P(E_s) = \sum_i t^i P_i(E_s),$$

where

$$P_i(E_s) = \sum_j t^j \operatorname{qdim}_{\mathbb{F}} E_s^{i,j}.$$

Lemma 5.2. *For any $s > 0$ the following equality holds, whenever it makes sense,*

$$P(E_s)(-1,q) = P(E_\infty)(-1,q).$$

Proof. This is a direct consequence of [McC01, Ex. 1.7]. □

Let us now analyze E_1 pages of the spectral sequences from Theorem 2.19, for odd p. Let us make the following notation. If $1 \leq i \leq p^n - 1$, then $i = p^{u_i} g$, where $\gcd(p,g) = 1$. Poincaré polynomials of columns are given below.

$$\begin{aligned}
P_0({}_{p^{n-s}}E_1) &= t^{c(D_{\alpha_0})} q^{3c(D_{\alpha_0})} \operatorname{KhP}_{p^n,p^{n-s}}(D_{\alpha_0}), && (1)\\
P_{p^n}({}_{p^{n-s}}E_1) &= t^{c(D_{\alpha_1})} q^{3c(D_{\alpha_1})+2p^n} \operatorname{KhP}_{p^n,p^{n-s}}(D_{\alpha_1}), && (2)\\
P_i({}_{p^{n-s}}E_1) &= \sum_{0\leq v\leq \min(s,u_i)} \sum_{\alpha\in\overline{\mathcal{B}}_i^{p^v}(X)} t^{c(D_\alpha)} q^{i+3c(D_\alpha)+p^n} \operatorname{KhP}_{p^v,p^1}(D_\alpha) + \\
&\quad + \sum_{\min(s,u_i)<v\leq u_i} \sum_{\alpha\in\overline{\mathcal{B}}_i^{p^v}} t^{c(D_\alpha)} q^{i+3c(D_\alpha)+p^n} \operatorname{KhP}_{p^v,p^{v-s}}(D_\alpha). && (3)
\end{aligned}$$

In order to make further computations more manageable let us introduce the following notation.

$$\begin{aligned}
G_i(v,w) &= \sum_{\alpha\in\overline{\mathcal{B}}_i^{p^v}} t^{c(D_\alpha)} q^{3c(D_\alpha)} \operatorname{KhP}_{p^v,p^w}(D_\alpha),\\
DJG_i(v,w) = G_i(v,w)(-1,q) &= \sum_{\alpha\in\overline{\mathcal{B}}_i^{p^v}} (-1)^{c(D_\alpha)} q^{3c(D_\alpha)} \operatorname{DJ}_{v,w}(D_\alpha)
\end{aligned}$$

so for $1 \leq i \leq p^n - 1$ the Poincaré polynomial can be expressed as the following more compact sum.

$$P_i({}_{p^{n-s}E_1}) = q^{i+p^n} \sum_{v=0}^{\min(s,u_i)} G_i(v,0) + \tag{4}$$

$$= q^{i+p^n} \sum_{v=\min(s,u_i)+1}^{u_i} G_i(v, v-s). \tag{5}$$

Lemma 5.3. *Following formula holds for the Poincaré polynomials*

$$\begin{aligned}
&P({}_{p^{n-s}}E_1) - P({}_{p^{n-s+1}}E_1) = \\
&\sum_{1 \leq j \leq p^{n-s}-1} t^{j\cdot p^s} q^{j\cdot p^s + p^n} \sum_{v=s}^{u_i} \left(G_{jp^s}(v, s-v) - G_{jp^s}(v, v-s+1)\right) \\
&+ t^{c(D_{\alpha_0})} q^{3c(D_{\alpha_0})+p^n} \left(\mathrm{KhP}_{p^n,p^{n-s}}(D_{\alpha_0}) - \mathrm{KhP}_{p^n,p^{n-s+1}}(D_{\alpha_0})\right) \\
&+ t^{c(D_{\alpha_1})+p^n} q^{3c(D_{\alpha_1})+2p^n} \left(\mathrm{KhP}_{p^n,p^{n-s}}(D_{\alpha_1}) - \mathrm{KhP}_{p^n,p^{n-s+1}}(D_{\alpha_1})\right).
\end{aligned}$$

Proof. Indeed, because

$$\begin{aligned}
&P_i({}_{p^{n-s}}E_1) - P_i(E_{p^{n-s+1}}E_1) = \\
&= \begin{cases} \sum_{v=s}^{u_i} \left(G_i(v, s-v)\right) - G_i(v, v-s+1)\right), & p^s \mid i, \\ 0, & p^s \nmid i, \end{cases}
\end{aligned}$$

which can be easily verified using formula (4). $\square$

Corollary 5.4. *The following formula holds for the difference polynomials*

$$\begin{aligned}
&\mathrm{DJ}_{n,n-s}(D) = \\
&\sum_{1 \leq j \leq p^{n-s}-1} (-1)^{j\cdot p^s} q^{j\cdot p^s + p^n} \sum_{v=s}^{u_i} DJG_{jp^s}(v, v-s) \\
&+ (-1)^{c(D_{\alpha_0})} q^{3c(D_{\alpha_0})+p^n} \mathrm{DJ}_{n-s}(D_{\alpha_0}) \\
&+ (-1)^{c(D_{\alpha_1})+p^n} q^{3c(D_{\alpha_1})+2p^n} \mathrm{DJ}_{n-s}(D_{\alpha_1}).
\end{aligned}$$

Proof. It follows easily from previous Lemma by substituting $t = -1$ and noting that $P({}_{p^{n-s}}E_1)(-1, q) = \mathrm{J}_{p^n,p^{n-s}}(D)$, by Lemma 5.2. $\square$

Definition 5.5. *For any $0 \leq i \leq p^n$ define a map*

$$\kappa \colon \mathcal{B}_i(X) \to \mathcal{B}_{p^n-i}(X)$$

$$\kappa(\beta)(c) = \begin{cases} 1 - \beta(c), & c \in X, \\ \beta(c), & c \notin X. \end{cases}$$

Proposition 5.6. *Let D be p^n-periodic link diagram and let $X \subset \mathrm{Cr}\, D$ be a chosen orbit of crossings. Suppose that all crossings from X are positive and let $D^!$ denote invariant link diagram obtained from D by changing all crossings*

from X to negative ones. Then the following equalities hold

$$\begin{aligned} D_\alpha &= D^!_{\kappa(\alpha)} \\ |\kappa(\alpha)|_u &= p^n - |\alpha|_u, \ \textit{for } u = 0, 1, \\ c(D_\alpha) &= c(D^!_{\kappa(\alpha)}) + p^n \end{aligned}$$

Proof. The first two equalities are direct consequences of definitions. To prove the third one notice that

$$n_-(D) = n_-(D^!) - p^n.$$

Therefore

$$\begin{aligned} c(D_\alpha) &= n_-(D_\alpha) - n_-(D) = n_-(D^!_{\kappa(\alpha)}) - n_-(D^!) + p^n = \\ &= c(D^!_{\kappa(\alpha)}) + p^n. \end{aligned}$$

□

Proof of Theorem 3.6. To prove the first part notice that from corollary 5.4 it follows that

$$\begin{aligned} \mathrm{DJ}_{n,0}(D) = &(-1)^{c(D_{\alpha_0})} q^{3c(D_{\alpha_0})+p^n} \mathrm{DJ}_{n,0}(D_{\alpha_0}) \\ &+ (-1)^{c(D_{\alpha_1})+p^n} q^{3c(D_{\alpha_1})+2p^n} \mathrm{DJ}_{n,0}(D_{\alpha_1}), \end{aligned}$$

because D_{α_0} and D_{α_1} are the only diagrams with isotropy group equal to $\mathbb{Z}/p^n$, because these are the only invariant diagrams.

Now without loss of generality assume that the chosen orbit of crossings consists of positive crossings. Then D_{α_0} inherits orientations from D and therefore $c(D_{\alpha_0}) = 0$. Therefore

$$\begin{aligned} \mathrm{DJ}_{n,0}\left(\ldots \right) = &\, q^{p^n} \mathrm{DJ}_{n,0}\left(\ldots \right) \\ &+ (-1)^{c(D_{\alpha_1})+p^n} q^{3c(D_{\alpha_1})+2p^n} \mathrm{DJ}_{n,0}(D_{\alpha_1}), \end{aligned}$$

On the other hand for $D^!$ as in the previous proposition $D^!_{\alpha_1}$ inherits orientation. Furthermore $c(D^!_{\alpha_1}) = -p^n$. This gives the following equality

$$\begin{aligned} \mathrm{DJ}_{n,0}\left(\ldots \right) = &(-1)^{c(D^!_{\alpha_0})} q^{3c(D^!_{\alpha_0})+p^n} \mathrm{DJ}_{n,0}(D^!_{\alpha_0}) \\ &+ q^{-p^n} \mathrm{DJ}_{n,0}\left(\ldots \right), \end{aligned}$$

Denote $c = c(D_{\alpha_1})$, then $D_{\alpha_1} = D^!_{\alpha_0}$ and $c(D^!_{\alpha_0}) = c - p^n$ by Proposition 5.6. Therefore

$$\begin{aligned} \mathrm{DJ}_{n,0}\left(\ldots \right) &= q^{p^n} \mathrm{DJ}_0\left(\ldots \right) \\ &\quad + (-1)^{c+p^n} q^{3c+2p^n} \mathrm{DJ}_{n,0}(D_{\alpha_1}), \\ \mathrm{DJ}_{n,0}\left(\ldots \right) &= (-1)^{c-p^n} q^{3c-2p^n} \mathrm{DJ}_{n,0}(\overline{D}_{\alpha_0}) \\ &\quad + q^{-p^n} \mathrm{DJ}_0\left(\ldots \right). \end{aligned}$$

From the above equalities the first part of the theorem follows easily.

To prove the second part, notice that Proposition 5.6 implies that for $v \geq s$ the following equality holds.

$$(-1)^i q^{i-p^n} DJG_i(v, v-s)(D) - (-1)^{p^n-i} q^{4p^n-i} DJG_i(v, v-s)(D^!) =$$
$$= \sum_{\alpha \in \overline{\mathcal{B}}_i^{p^v}} (-1)^{c+p^n+1} q^{3c+3p^n} \, \mathrm{DJ}_{v,v-s} \left(q^{i-p^n} - q^{p^n-i} \right) (D_\alpha).$$

Consequently, if $p^s \mid i$, then the above difference is divisible by $q^{p^s} - q^{-p^s}$. To finish the proof combine formula 5.4 and with the discussion above. $\square$

5.2 Proof of Theorem 4.1

Proof of Theorem 4.1. According to the definition, the equivariant Jones polynomials, can be written as the following sum.

$$q^{-n_+(D)+n_-(D)} \, \mathrm{J}_{p^n, p^{n-s}}(D) =$$
$$= \sum_{r=n_-(D)}^{n_+(D)} (-q)^r \, \mathrm{qdim}_{\mathbb{Q}[\xi_{p^{n-s}}]} \mathrm{Hom}_{\mathbb{Q}[\mathbb{Z}/p^n]} \left(\mathbb{Q}\left[\xi_{p^{n-s}}\right], \mathrm{CKh}^{r-n_-(D),*}(D) \right). \quad (6)$$

Thus, the only thing we need to do is to determine the quantum dimension of the following graded module

$$\mathrm{Hom}_{\mathbb{Q}[\mathbb{Z}/p^n]} \left(\mathbb{Q}\left[\xi_{p^{n-s}}\right], \mathrm{CKh}^{r-n_-(D),*}(D) \right)$$

for $0 \leq r \leq n_+(D)+n_-(D)$. Performing calculations as in the proof of Theorem 2.19. Let $r = p^{u_r} g$, where $\gcd(p, g) = 1$. We obtain the following formula.

$$\mathrm{qdim}\, \mathrm{Hom}_{\mathbb{Q}[\mathbb{Z}/p^n]} \left(\mathbb{Q}\left[\xi_{p^{n-s}}\right], \mathrm{CKh}^{r-n_-(D),*}(D) \right) =$$
$$= \sum_{v=0}^{\min(s,u_r)} \sum_{s \in \mathcal{S}_r^{p^v}(D)} \mathrm{J}_{p^v,1}(s) + \sum_{v=\min(s,u_r)+1}^{u_r} \sum_{s \in \mathcal{S}_r^{p^v}} \mathrm{J}_{p^v, p^{v-s}}(s).$$

Plugging the above formula into (6) and taking the difference

$$\mathrm{J}_{p^n, p^{n-s}}(D) - \mathrm{J}_{p^n, p^{n-s+1}}(D)$$

yields the desired formula. $\square$

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